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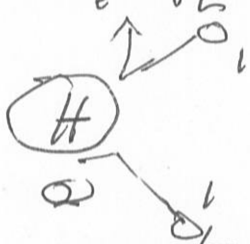
Transport II — Scaling and Scales in Kinetics

→ Here continue transport:
- "slow processes" → deflection, stopping
- long lmfp
- transport in weakly ionized systems
- plasma transport — intro.

(i) Weak Deflection

- How do energy and momentum change in 'slow processes' i.e. where there is a weak deflection of a quantity in each collision.. Point is many kicks accumulate!

Now, consider trace heavy in gas of lights → mean square momentum change? deflection?
→ time to fully deflect?



In collision, by momentum conservation:

$$\Delta p_2 \sim \Delta p_1$$

(Heavy stationary frame) and light momentum change large

i.e. $\Delta p_1 \sim p_1$ so

(i.e. light bounces off)

Recall:

→ 'stuff'

→ transport

→ response / linear

i.e. $$\Gamma = -D \nabla n$$

Γ ↓ flux ∇n ↓ force

→ $$dS/dt \sim D(\nabla n)^2 > 0$$

$$\sim -\Gamma \nabla n$$

$$\underline{V} \cdot \underline{F}$$

→ various cases dilute gas:

$$l_{mfp} \sim 1/n\sigma \qquad d < \bar{r} < l_{mfp} < L$$

→ $$D \sim v_{th}\, l_{mfp}$$

→ heavy on light gas:

See:

– Kraichnan, 'conclusion' for perspective

– HW.

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$\langle (\Delta p_{2\perp})^2 \rangle \sim p_1^2$

for deflection length ?:

then: $\frac{d}{dl}\langle (\Delta p_{2\perp})^2 \rangle \sim \frac{p_1^2}{l_{mfp}} \sim p_1^2 (n\sigma)$

$n = n_L$
$\sigma = \sigma_{LH}$
$v_{th} = v_{th_L}$

for deflection time ?:

$$\frac{d}{dt}\langle (\Delta p_{2\perp})^2 \rangle \sim v_{th_L} n\sigma\, p_1^2$$

[Lights mediate deflection]
↓
Lights dominate relative speed]

so

$$\boxed{\langle (\Delta p_{2\perp})^2 \rangle \sim (v_{th_L} n\sigma\, p_1^2)\, t}$$

i.e.

$$\langle (\Delta p_{2\perp})^3 \rangle \sim D_p\, t$$

momentum space diffusivity

$\Rightarrow$

$$\boxed{D_{p_2} = p_1^2 (n\sigma v_{th_L})}$$

$\Delta p_2 \sim p_1$
$\Delta t \sim 1/n\sigma v_{th_L}$
$\sim \nu_{c_{LH}}^{-1}$

For mobility of heavy? → HW
Spatial diffusion of heavy?

For deflection angle;

$\Delta P_2 \sim P_2 \Delta\Theta$ i.e. small kick of P_2 $\Rightarrow \Delta\Theta$

so:

$$\boxed{\langle(\Delta\Theta)^2\rangle \simeq v_{th}\sigma n\, P_1^2/P_2^2\, t}$$

$$\sim n\sigma\sqrt{\frac{T}{M_1}}\;\frac{M_1^{\cancel{2}}\,\cancel{T/M_1}}{M_2^{\cancel{2}}\,\cancel{T/M_2}}\;t$$

angular deflection of time $\sim \left(n\sigma\sqrt{T}\,\sqrt{M_1}/M_2\right)t$ $\Rightarrow$ deflection angle diffuses.

$\tau_{scatt} \sim M_2/n\sigma(TM_1)^{1/2}$ $\rightarrow$ higher temp randomizes faster.

How many collisions to deflect?

Also $\tau_c/\tau_{scatt} \sim \left(\frac{1}{\cancel{n\sigma} v_{th}}\right)\frac{\cancel{n\sigma}(TM_1)^{1/2}}{M_2}$

$$\boxed{\tau_c/\tau_{scatt} \sim M_1/M_2.}$$

Randomization / full deflection occurs after $M_2/M_1 \gg 1$ collisions!

What of Energy?

⇝ attack ⊙ same way:

$$E_2 \sim P_2^2/2M_2 \Rightarrow \Delta E_2 \sim P_2 \Delta P_2/M_2$$

$$\Delta P_2 \sim P_1$$

so

$$\Delta E_2 \sim P_2 P_1/M_2 \sim \sqrt{M_1/M_2}\, T \ll T \sim E_2$$

(equal T)

$$\therefore \quad \Delta E_2 \ll E_2$$

$$\Rightarrow \langle(\Delta E_2)^2\rangle \simeq \langle(\Delta E_2)^2\rangle_{\text{coll.}}\, n\sigma v_1 t$$

$$\boxed{\langle(\Delta E)^2\rangle \simeq \left(\frac{M_1}{M_2} T^2\right) n\sigma v_{th,1} t}$$

$$\boxed{D_E \sim \frac{M_1}{M_2} T^2 n\sigma v_{th,1}}$$ ⟺ diffusivity for energy ~~[illegible]~~ randomization

For complete randomization:

$$\langle(\Delta E_2)^2\rangle \sim E_2^2 \sim T^2$$

$$\tau^E_{scatt} \sim M_2/\sqrt{M_1 T}\, n\sigma$$

(c) Long → Now: Mean Free Path

Key to kinetics:

$$d < \langle n\rangle^{-1/3} < l_{mfp} < L$$

fundamental ordering

Knudsen # $\sim l_{mfp}/L$.

What if $l_{mfp} > L$? → i.e. rarefied gas

→ "long mean free path".

n.b.: – $l_{mfp} \sim 1/n\sigma$, so rarefied → low n

– $\bar{r} < L$ → can think kinetically!...

Now, how approach transport in long l_{mfp} regimes?

$K_n < 1$ → usual, transport is local

i.e. $q = -\chi \nabla T$

$\chi = cD$, where $\chi = \chi(x)$ → "local"

n.b.

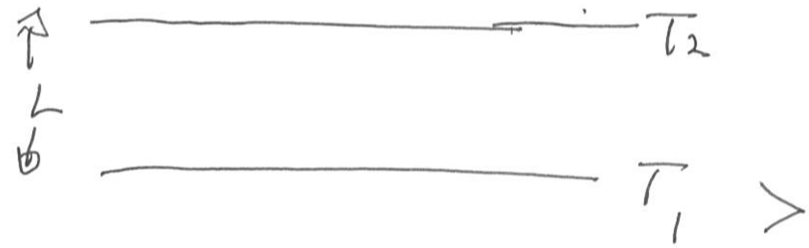
– χ is intensive, not extensive, depends only on local thermodynamic quantities, on l_{mfp} scale. See HW problem.

n.b. l_{mfp} is minimum resolvable scale in hydrodynamics..

For $K > 1$

— transport depends upon macroscopics / configuration of system, i.e. L scale (boundaries)

i.e.



what is heat flux from

What is heat flux from ① → ② ?

Here $\Delta T < T_1, T_2$ so v_{Th} meaningful.

Now, argue:

— net flux thru gas = flux energy in at ① − flux energy out at ②

$$-q = -\chi \nabla T \rightarrow \chi \frac{(T_1 - T_2)}{L}$$

χ changes.

Simply, # "collisions" / events to one wall / area

$$\sim n v_{Th}$$

so

$$q \approx \frac{1}{A}\left(A n v_T (T_1 - T_2)\right)$$

energy in to gas

energy out of gas

so

so, for lmfp large:

$$\boxed{\lambda \cong n v_{Th} L}$$ → effective thermal conductivity

- L replaces lmfp → macroscale plays role of mean free path
- generally: $\lambda = n v_{th} l_{min}$

$$l_{min} = \min[l_{mfp}, L]$$

or

$$\boxed{\lambda \simeq n D \Big/ \underbrace{\left(1 + (l_{mfp}/L)^2\right)^{1/2}}_{\text{"flux limiting" factor}}}$$

often referred to as flux-limited diffusion

- $\boxed{\lambda \sim P L / \sqrt{MT}}$ $\qquad P \equiv$ pressure

Aside: (general culture)

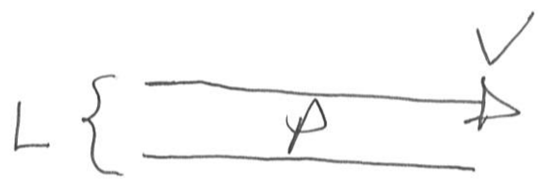
Long lmfp

$$Q = -\chi \nabla T \rightarrow -\int dx' \underbrace{K(x, x')}_{\text{kernel}} \nabla T(x')$$

→ $K \to \chi \delta(x - x')$ ⇒ local limit

Some Applications (More):

→ consider two parallel plates in relative motion, at velocity V i.e. move upper.



gap

[redacted] $L < l_{mfp}$.

What is force on plate?

$$\frac{F}{A} = -\eta \, \nabla V \rightarrow \eta V / L$$

motion, opposing the motion.

As above, $\eta = \rho \nu = m n v_{th} l_{mfp}$

$$\rightarrow m n v_{th} L$$

$$\sim p L \sqrt{m/T}$$

so, if compare to pressure:

$$F/A/p \sim \frac{p \cancel{L} \sqrt{m/T} \, V/\cancel{L}}{p} \sim V/v_{th} \ll 1$$

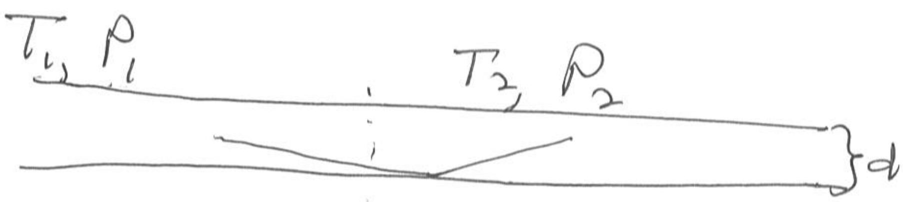
tiny.

For F, $F \sim A p \, V/v_{th}$. [redacted]

→ Knudsen Problem

i.e. Knudsen problem

Consider tube of rarefied gases, constructed by placing 2 tubes adjacent and removing barrier, i.e.



$d << l_{mfp}$.

- Nature of equilibrium when barrier removed? Pressure balance? Knudsen #
- Point is that $K \sim l_{mfp}/d > 1 \Rightarrow$ particles collide more frequently with wall than each other
- temps $\to$ wall.
- Now have:

$$n_1, v_1 \qquad\qquad n_2, V_2$$

Fluxes must balance:

$$n_1 v_1 d^2 \sim n_2 V_2 d^2$$ ~ i.e. continuity and stationarity set up form.

so

$$P_1/\sqrt{T_1} \sim P_2/\sqrt{T_2}$$ ~ T set by wall.

Interestingly, $P_1 \neq P_2$ possible $\Rightarrow$

iii.) Weakly Ionized Gas

i.e. what is it?

~ consider a weakly ionized gas

i.e. $n_e \sim Z n_i \ll n_n \sim n$.

i.e. most particles neutral. So, basic picture of system is:

- 'heavy' particles of neutrals
- fewer, 'light' electrons

neutral scattering ⇒ finite range

~ New Transport Coefficient - Conductivity

i.e. seek σ, s/t $\underline{J} = \sigma \underline{E}$, for weak $\underline{E}$.

Now $\underline{J} = n_0 e \underline{V}$ $\qquad e = -q$

$\underline{V} = \mu \underline{F} = \mu e \underline{E}$

force

$\therefore \quad \underline{J} = n_0 e^2 \mu \underline{E}$

So conductivity set by electron mobility

Now, $\mu = l_{mfp}/p$ $\qquad$ i.e. $\mu \sim 1/(n \sigma v_{th})$.

$l_{mfp} = l_{mfp}$ e,n collisions $\qquad \sim 1/$

So

$$\underline{v} = \frac{n_0 e^2 \underline{l_{mfp}}}{p_e} \underline{E}$$

$$\boxed{\sigma_e = n_0 e^3 l_{mfp}/p_e}$$

is conductivity for weak E.

Show Chapman-Enskog derivation.

- or:

$$\sigma_e = (c_e e^3 l_{mfp}/p_e)\, n_1 \sim n_e/n \to \text{electron concentration}$$

$$= c_e e^2 \frac{1}{n_1 \sigma\, p_e} \cancel{n} \sim c_e e^2/\sigma\sqrt{mT}$$

- linear response, $\Rightarrow \sigma_e$ indep. E.

- $\Delta E_{\substack{single \\ colln}} \sim (m/M)^{1/2} T$

then: $\dfrac{\tau_{relax}}{\tau_c} \sim \dfrac{M}{m} \dfrac{l}{v_{th}} \to$ i.e. M/m collns to scatt. energy.

$\tau_{relax}: \langle(\Delta E)^2\rangle \sim E^2$

What of strong field?:

- ΔE in $l_{mfp} \sim e\tilde{E}\, l_{mfp}$

- as can add randomly:

$$(\Delta E)^2 \sim M/m\, (e\tilde{E}\, l_{mfp})^2 \text{ after } M/m \text{ collns.}$$

so $(\Delta E)^2 > T^2$

Point is that after many collisions $\langle E\rangle|_{E} > T$

- $\underset{\sim}{\epsilon} \sim \sqrt{M/m}\,(e\tilde{E}\, l_{mfp})$

– For drift speed, i.e. not random:
increment

$\Rightarrow \frac{1}{2} M V^2 \sim \mathcal{E} \Rightarrow$ average velocity

$$V \sim \sqrt{\mathcal{E}/M} \sim (eE\, l_{mfp})^{1/2} (m/M^3)^{1/4}$$

$\Rightarrow$ Then:

if $\mathcal{E} \to \mathcal{E} + \Delta\mathcal{E}$ energy changes by $\Delta\mathcal{E}$
then

$\Rightarrow \Delta V \sim \Delta\mathcal{E} / \sqrt{M\mathcal{E}}$ – increment!
(about $\mathcal{E}$ energized state).

$\Rightarrow$ in 1 l_{mfp}; $\Delta\mathcal{E} \sim eE\, l_{mfp}$
$\therefore$

$$\Delta V \sim eE\, l_{mfp} / \sqrt{M\mathcal{E}}$$ increment %.

$$\Rightarrow \Delta V \sim (eE\, l_{mfp})^{1/2} / \sqrt[4]{mM}$$

$\therefore$ drift velocity:

$$\boxed{V_d \simeq \Delta V \simeq (eE\, l_{mfp})^{1/2} / (mM)^{1/4}}$$ nonlinear

$\Rightarrow$ now, for conductivity:

For $\underline{J} = \sigma_e \underline{E}$

$$\boxed{\sigma_e = n e^2 \Big/ (mM)^{1/4} \sqrt{eE n\sigma}}$$

to see, take:
$\sigma_e \sim n_e \frac{l_{mfp} e^2}{p_e}$

and:
$p_e \sim m_e v$
$v \sim (eE l_{mfp})^{1/2} (M/m^3)^{1/4}$

– nonlinear conductivity in strong field limit.

$\sim 1/\sqrt{E}$

point is E sets V
$\Rightarrow \sigma$ develops NL.

N.B. Chapman – Enskog Approach:

Conductivity: (Linear, only).

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f + \frac{e\underline{E}}{m} \frac{\partial f}{\partial \underline{v}} = C(f)$$

$$\frac{e}{m} E \frac{\partial f}{\partial v} = -\nu_{gn} \delta f$$

l.o. $\quad f = f_0^M$

1st

$$\frac{e}{m} \underline{E} \cdot \frac{\partial f_0}{\partial \underline{v}} = -\nu_{gn} \delta f$$

$$\underline{J} = \sigma \underline{E}$$

then

→

$J \sim \dfrac{n_0 |e|^2 v_{th} E_0}{m_e v_{th} \nu}$ $\qquad$ $\sigma_e \sim \dfrac{n_0 e^2 l_{mfp}}{m_e \, v_{th}}$

so checks ✓ $\qquad$ ✓

iii.) Plasmas → what are it?

- gas, dilute
- charged particles, ionized but net neutral
- Coulomb, long range interaction.

→ return to basics!

- force is Coulomb → scale free
  - screens

∴

- no ℓ → no intrinsic scale force.
- Coulomb force is long range ⇒ glancing collisions.

so

$d < n^{-1/3} < l_{mfp} < L$

⇒ now:

$$\boxed{n^{-1/3} < \lambda_D < l_{mfp} < L}$$

screening!

- λ_D → Debye Length new!

i.e.

in plasma.

- q_T

$+ \rightarrow +$ T here sees zip

$+ \rightarrow q_T$

$+ \; - \; +$

i.e. plasma charges adjust to screen

$1/r \rightarrow e^{-r/\lambda_D}/r$ test:

$$\nabla^2\phi = -4\pi\rho$$

$$= -4\pi n_0 |e| \left[\delta n_i - \delta n_e\right] + 4\pi q_T \delta(x - x_T)$$

$\Rightarrow$

$$\delta n_i = n_0 \exp\left[-\frac{e\phi}{T_i}\right]$$

$$\delta n_e = \exp\left[+\frac{e\phi}{T_e}\right]$$

so noting neutrality:

$$\nabla^2\phi = 4\pi n_0 e^2 \left[\cancel{1} + \frac{e\phi}{T_i} - \cancel{1} + \frac{\phi}{T_e}\right]$$

$$\nabla^2\phi = 4\pi n_0 e^2 \left(\frac{1}{T_e} + \frac{1}{T_i}\right)\phi$$

$$\boxed{1/\lambda_D^2 \cong 4\pi n_0 e^2 \left(1/T_e + 1/T_i\right)}$$

screening;

$\boxed{\text{Debye Length.}}$

$\lambda_D^2 \approx [4\pi n_0 e^2 \ldots$

Key feature of plasma:

$n\lambda_D^3 > 1 \Rightarrow \lambda_D > \bar{r}$

[really needed to screen]

$\bar{r} \sim \langle n \rangle^{-1/3} \rightarrow$ mean inter-part. spacing

~ many particles in Debye sphere

~ why? $T > e^2/\bar{r}$ diluteness!

Debye sphere $\Leftrightarrow$ Diluteness same.

$\frac{Tn}{ne^2}\bar{r} > 1 \Rightarrow \lambda_D^2 \bar{r} n > 1$

$\Rightarrow \lambda_D^2 > \bar{r}^2$!

Also – plasma classical:

$T \gg \epsilon_{quant.} \sim p^2/2m \sim \hbar^2/\bar{r}^2 2m)$ from Heisenberg

quantum energy (smallest scale)

$\Rightarrow \boxed{T \gg \hbar^2 n^{2/3}/m}$

and had:

$T \gg e^2 n^{1/3}$

[i.e. diluteness criterion stronger than quant.]

$\Rightarrow$ $e^2 \; n^{1/3}$ ... 2

where

$$a_B = 4\pi\hbar^2/me^2 \rightarrow \text{Bohr Radius}$$

$a_B \sim 1\,nm$.

$\Rightarrow$ conditions for plasma (classical):

$$\boxed{\begin{array}{c} n\lambda_D^3 >> 1 \\ \lambda_D^3/a_B^3 >> 1. \end{array}}$$

and scale ordering:

$$\boxed{\bar{r} < \lambda_D < l_{mfp} < L}$$ for plasma.

Frequencies / Resonances

$$\underline{\nabla} \cdot \underline{D} = 4\pi \rho_{ext}$$

$$\underline{D} = \underline{E} + 4\pi \underset{\text{polarization}}{\underline{P}} = \overset{\text{dielectric fctn}}{\epsilon(\omega)}\, \underline{E}$$

and, say, electron polarization

Consider high ω wave $\rightarrow$ electron osc

inertia low!

$$m_e \frac{d^2 \underline{x}}{dt^2} = e \underline{E}$$

$$-\omega^2 \underline{x} = e \underline{E} \quad \Rightarrow \quad \delta \underline{x} = -\frac{e \underline{E}}{\omega^2}$$

so $4\pi \underline{P} = \frac{4\pi n_0 e^2}{m_e \omega^2} \underline{E} = \frac{\omega_{pe}^2}{\omega^2} \underline{E}$

$$\boxed{\omega_{pe}^2 = \frac{4\pi n_0 e^2}{m_e} \rightarrow \text{plasma frequency}}$$

$\rightarrow$ space charge oscillation wave

$\rightarrow \delta n \rightarrow \delta E \rightarrow$ restoring

Gauss Law

so

$$\underline{D} = \left(1 - \frac{\omega_p^2}{\omega^2}\right) \underline{E}$$

$$\epsilon(\omega) = 1 - \omega_p^2/\omega^2$$

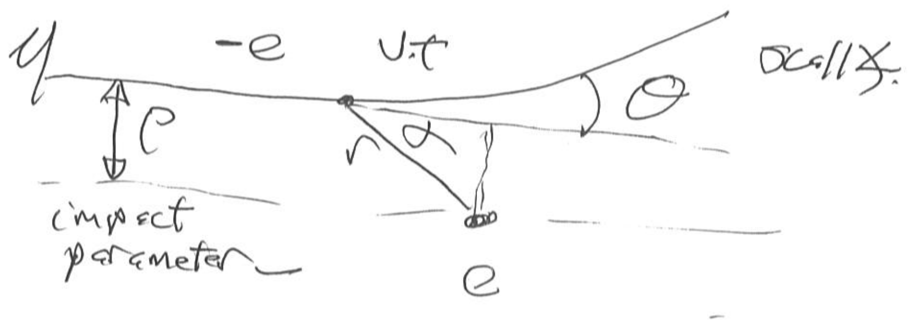
$$\underline{\nabla} \cdot \underline{D} = \left(1 - \frac{\omega_p^2}{\omega^2}\right) \underline{\nabla} \cdot \underline{E} = 4\pi \rho_{ext}$$

so, for $\rho_{ext} = \rho_{ext}(\omega \sim \omega_{pe})$,

$\underline{E}$ response in plasma is large

Collisions – Coulomb $\Leftrightarrow$ Transport.

Consider familiar collision:



– what is cross section?
– no $d\sigma$
seek momentum transfer cross section

– more glancing collns.
$\Rightarrow$ range

$p = \mu v$

deflection:

$$M \Delta V_\perp \to \Delta p_\perp = \int_{-\infty}^{+\infty} dt\, F_\perp$$

$$= \int_{-\infty}^{+\infty} dt\, \frac{e^2}{r^2} \sin\alpha$$

$\Rightarrow$

$$\Delta p_\perp = e^2 \int_{-\infty}^{+\infty} \frac{\rho\, dt}{(\rho^2 + v^2 t^2)^{3/2}} \sim e^2/\rho v$$

so

$$\boxed{\theta \approx e^2/\mu v^2 \rho} \to \text{deflection angle}$$

$\Rightarrow$

$$d\sigma \sim \rho\, d\rho \sim d\left(\frac{e^2}{\mu v^2 \theta}\right)^2$$

area of

Now, for momentum transfer cross-section (transverse) takes out head-on.

$$d\sigma_t \simeq (1-\cos\theta)\, d\sigma \simeq \left(\frac{e^2}{\mu v^2}\right)^2 \frac{d\theta}{\theta}$$

$\int d\sigma_t \Rightarrow$

$$\Rightarrow \boxed{\sigma_t \sim \left(e^2/\mu v^2\right)^2 \ln\left(1/\theta_0\right)}$$

$\rightarrow$ divergence, $\ln\theta$!

- Coulomb cross-section (Rutherford)
- θ_0 is small angle cut-off.

What sets θ_0 $\Leftrightarrow$ what is weakest angle? Screening!

Now $\theta \sim e^2/\mu v^2 \rho$ i.e. θ small $\rightarrow$ large impact!

ρ: impact parameter.

Now, $\rho > \lambda_D$ screened! $\rightarrow$ don't see Coulomb force.

λ_D sets θ_0

$\Rightarrow$ $\theta_0 \sim e^2 / \ldots$

$\rightarrow \ln \Lambda = \ln(1/\theta_0)$

$= \ln\left(\frac{T\lambda_D}{e^2}\right) \rightarrow$ Coulomb Logarithm

$$\boxed{\sigma_t \sim \frac{e^2}{T}\ln\Lambda} \rightarrow \text{effective cross-section}$$

$\therefore$ $$\boxed{\sigma_t \sim \bar{r}^2\left(e^2/\bar{r}T\right)^2 \ln\Lambda} \rightarrow \text{effective Coulomb cross-section}$$

Note: $\underbrace{\left(e^2/\bar{r}T\right)^2}_{\bar{r}^2/\lambda_D^2} \rightarrow \left(\left[1/n\lambda_D^3\right]^{2/3}\right)^2 \sim \frac{\bar{r}^4}{\lambda_D^4}$

$\sim \left(1/n\lambda_D^3\right)^{4/3}$

$$\left\{\sigma_t \sim \bar{r}^2\left(1/n\lambda_D^3\right)^{4/3}\ln\Lambda\right.$$

$\Rightarrow$

$l_{mfp} \sim 1/n\sigma_t \sim 1\Big/n\bar{r}^2\left(\frac{e^2}{\bar{r}T}\right)^2\ln\Lambda$

$\sim \bar{r}\left(n\lambda_D^3\right)^{4/3}/\ln\Lambda$

$$l_{mfp} \sim \bar{r}\,(\lambda_D/\bar{r})^4 / \ln \Lambda$$

so

$$\frac{l_{mfp}}{\lambda_D} \simeq \frac{\bar{r}}{\lambda_D}\left(\frac{\lambda_D}{\bar{r}}\right)^4 \Big/ \ln \Lambda$$

$$\simeq (\lambda_D/\bar{r})^3 / \ln \Lambda$$

$$\cong (n\,\lambda_D^3) / \ln \Lambda$$

i.e. $n\,\lambda_D^3 \gg \ln \Lambda$

so $\quad l_{mfp}/\lambda_D \gg 1 \quad \Rightarrow \Big\{$ establishes consistency with screening

Note:

- apart from $\ln \Lambda$, no mass scaling in σ_t, l_{mfp}

$\therefore$

- $\tau_{colln} \sim (M)^{1/2}$ short → electrons!

so $\tau_{e,e} / \tau_{i,i} \sim (m/M)^{1/2} \ll 1$.

- as before, have:

$$\underset{\substack{\text{energy} \\ \text{exchange} \\ e, i}}{\tau_{e,i}} \sim (M/m)\, \tau_{e\,\text{colln}}$$

⇒ energy exchange <u>much</u> slower than collisional equilibration of each species individually.

- as before:

→ thermal conductivity:

$$\lambda \sim n\, v_{th}\, l_{mfp}$$

↑ larger for electrons